Moduli theory of v. bundles and Gieseker stability.

Recall:

X smth proj. var. $/k$

The functor

$$\mathrm{Sch}_k^{op} \longrightarrow \mathrm{Sets}$$
$$S \longmapsto \mathrm{Pic}(X \times S)/p_S^*\mathrm{Pic}(S) \quad \text{where } p_S: X \times S \to S.$$

is represented by the Picard scheme Pic_X, i.e.,

for any $S \in \mathrm{Sch}_k$, $\quad \mathrm{Pic}(X \times S)/p_S^*\mathrm{Pic}(S) = \mathrm{Hom}_k(S, \mathrm{Pic}_X) =: \mathrm{Pic}_X(S)$

In particular, we have $\underbrace{\mathrm{Pic}_X(k)}_{\text{closed pts of } \mathrm{Pic}_X} = \mathrm{Pic}(X)$

However, Pic_X can be huge, but it decomposes into small pieces:

$\mathrm{Pic}_X = \coprod \underbrace{\mathrm{Pic}_X^P}_{\text{projective}}$, where $\mathrm{Pic}_X^P(k)$ parametrizes line bundles on X w/ a fixed Hilb. polynomial P.

Recall by HRR thm, we compute

$$P(\mathcal{L}, m) := \chi(X, \mathcal{L}(m))$$
$$= \int ch(\mathcal{L}) \cdot ch(\mathcal{O}(m)) \cdot td(X),$$

which only depends on $c_1(\mathcal{L})$ and $(X, \mathcal{O}(1))$.

<u>Question</u>. Can we construct moduli theory for v. bundles of higher ranks?

Two problems even on $\mathbb{P}^1$.

(finite type)

Consider $E_n := \mathcal{O}(n) \oplus \mathcal{O}(-n)$ on $\mathbb{P}^1$ $(n>0)$. Clearly, $h^0(E_n) = n+1$, so each E_n is not isom. to each other. On the other hand, $c_i(E_n) = 0$ $\forall i>0$, so they all share the same Hilb. polynomial (and the same Mukai vectors).

Now, suppose there is a moduli space $\mathcal{M}$ parametrizing all E_n.

Then, since $h^0(E) \geq n$ is a closed condition, $\mathcal{M}$ has a strictly descending chain $\cdots \subsetneq \underset{\substack{\cup\!\!\!\!\cdot \\ E_n}}{Z_n} \subsetneq \cdots \subsetneq Z_0 = \mathcal{M}$, so $\mathcal{M}$ is not of finite type.

(separatedness)

On $\mathbb{P}^1$, $\mathrm{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(1), (-1)) = H^1(\mathbb{P}^1, \mathcal{O}(-2)) = k$,

so $k \ni \lambda \xleftrightarrow{1:1} 0 \to \mathcal{O}(-1) \to E_\lambda \to \mathcal{O}(1) \to 0$. (Note $E_\lambda \cong \mathcal{O} \oplus \mathcal{O}$ if $\lambda \neq 0$).

We can then construct a v. bundle E on $\mathbb{A}^1 \times \mathbb{P}^1$ so that $E|_{\lambda \times \mathbb{P}^1} \cong E_\lambda$

Indeed, for v. bundles $\mathcal{F}, \mathcal{G}$ on X, we can construct a v. bundle $\mathcal{E}$ on $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \times X$

so that $\mathrm{Ext}^1(\mathcal{F}, \mathcal{G}) \ni \lambda \xleftrightarrow{1:1} 0 \to \mathcal{F} \to \mathcal{E}|_{\lambda \times X} \to \mathcal{G} \to 0$. "universal family of extensions".

Rmk 3.5.

Suppose there exists a moduli space $\mathcal{M}$ of higher rank v. bundles on $\mathbb{P}^1$,
i.e., $\mathrm{Hom}(X, \mathcal{M}) = \{\text{v. bundles on } X \times \mathbb{P}^1\}/\sim$. Then, E above induces a morphism
$\mathbb{A}^1 \to \mathcal{M}$ such that $\mathbb{A}^1 \setminus \{0\}$ gets sent to $[\mathcal{O} \oplus \mathcal{O}] \in \mathcal{M}_k$ and $\{0\}$ gets sent to $[\mathcal{O}(1) \oplus \mathcal{O}(-1)]$
$\rightsquigarrow \mathcal{M}$ is not separated. (consider base change to each spec k in $\mathbb{A}^1$).

Thus, we should add some extra conditions. Instead of slope stability,
we'll use (Gieseker) stability in $\dim > 1$, which has the following advantages.

(i) We have more pts in the "moduli space".

(ii) translation to GIT-stability is more direct.

(iii) Stability for torsion sheaves makes sense.

Recall. $(X, \overset{H}{\overset{\shortparallel}{\mathcal{O}(1)}})$ polarized proj. scheme.

The Hilb. polynomial for a coh. sheaf $\mathcal{F}$ is

$$P(\mathcal{F}, m) := \chi(\mathcal{F}(m)) = \sum_{i=0}^{d} \alpha_i(\mathcal{F}) \frac{m^i}{i!} \quad \text{w/ } \alpha_i(\mathcal{F}) \in \mathbb{Z} \text{ and } d = \dim(\operatorname{supp} \mathcal{F}) =: \dim \mathcal{F}.$$

Note if $\dim \mathcal{F} = \dim X = d$, then $\mathrm{rk}\, \mathcal{F} = \dfrac{\alpha_d(E)}{\alpha_d(\mathcal{O})}$.

A coh. sheaf E of dim d is called pure if for any subsheaf $F \subseteq E$, $\dim F = d$.

A sheaf of max. dim. is pure $\Leftrightarrow$ tors.-free.

Def. The reduced Hilb. poly. of a sheaf E of $\dim = d$ is $p(E,m) := \frac{P(E,m)}{\alpha_d(E)}$.

A coh. sheaf E is stable $\Leftrightarrow$ E is pure and

$$p(F,m) < p(E,m) \quad m \gg 0 \quad {}^{\forall} 0 \neq F \subsetneq E.$$

$$(\Leftrightarrow \ p(F,m) < p(E,m) \ )$$

lexicographic order on coeff's

" semi-stable $\Leftrightarrow$ $p(F,m) \leq p(E,m)$ "

E.g. X surface. $\mathcal{F} \in \mathrm{Coh}(X)$.

On a surface, $P(\mathcal{F},m) = \frac{\mathrm{rk}\,\mathcal{F}.(H)^2}{2} m^2 + (\underbrace{H.c_1(\mathcal{F})}_{\deg_H \mathcal{F}} + \mathrm{rk}\,\mathcal{F}(\underbrace{H.c_1(X)}_{\deg_H(T_X)}))m + \alpha_0(\mathcal{F})$ by HRR.

$\mathrm{rk}\,\mathcal{F} \neq 0$

$$\rightsquigarrow \ p(\mathcal{F},m) = \frac{m^2}{2} + \frac{m}{(H^2)}\left(\underbrace{\frac{\deg_H \mathcal{F}}{\mathrm{rk}\,\mathcal{F}}}_{``\mu_H(\mathcal{F})"} + \deg_H T_X\right) + \frac{\alpha_0(\mathcal{F})}{\mathrm{rk}\,F\,(H)^2}.$$

(i) $\dim \mathcal{F} = 0 \rightsquigarrow P(\mathcal{F},m) \equiv \mathrm{const} \rightsquigarrow p(\mathcal{F},m) = 1$.

Clearly, $\mathcal{F}$ is pure and semi-stable.

$\mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F} = k(x)$ for some $x \in X_k$

(ii) $\dim \mathcal{F} = 1 \rightsquigarrow \mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F}$ pure and ${}^{\forall} 0 \neq \mathcal{G} \subsetneq \mathcal{F}$,

$(\mathrm{rk}\,\mathcal{F} = 0)$

$$\deg_H(\mathcal{F}) > \deg_H(\mathcal{G}) \ \text{or} \begin{cases} \deg_H(\mathcal{F}) = \deg_H(\mathcal{G}) \\ \alpha_0(\mathcal{F}) > \alpha_0(\mathcal{G}). \end{cases}$$

If $\mathcal{F}$ is supported on an irr. curve C, then $\Leftrightarrow$ $\mathcal{F}|_C$ is μ-stable.

(iii) $\dim \mathcal{F} = 2 \rightsquigarrow \mathcal{F}$ is stable $\Leftrightarrow$ $\mathcal{F}$ is tors. free and ${}^{\forall} 0 \neq \mathcal{E} \subsetneq \mathcal{F}$.

$$\mu_H(\mathcal{F}) > \mu_H(\mathcal{E})$$

$$\text{or} \begin{cases} \mu_H(\mathcal{F}) = \mu_H(\mathcal{E}) \\ \frac{\alpha_0(\mathcal{F})}{\mathrm{rk}(\mathcal{F})} > \frac{\alpha_0(\mathcal{E})}{\mathrm{rk}\,\mathcal{E}} \end{cases}$$

Cor. X surface, $\mathcal{F}$ tors. free

μ-stable $\Rightarrow$ stable $\Rightarrow$ semi-stable $\Rightarrow$ μ-semistable.

Prop. (cf. 10.1.6.) A semi-stable sheaf has a JH filtration.